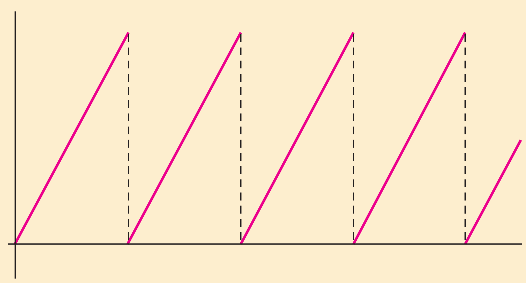
7 THE LAPLACE TRANSFORM



In the linear mathematical models for a physical system such as a spring/mass system or a series electrical circuit, the right-hand member, or input, of the differential equations

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t) \qquad \text{or} \qquad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

is a driving function and represents either an external force $f(t)$ or an impressed voltage $E(t)$. In Section 5.1 we considered problems in which the functions f and E were continuous. However, discontinuous driving functions are not uncommon. For example, the impressed voltage on a circuit could be piecewise continuous and periodic such as the “sawtooth” function shown above. Solving the differential equation of the circuit in this case is difficult using the techniques of Chapter 4. The Laplace transform studied in this chapter is an invaluable tool that simplifies the solution of problems such as these.

7.1 DEFINITION OF THE LAPLACE TRANSFORM

REVIEW MATERIAL

- Improper integrals with infinite limits of integration
- Partial fraction decomposition

INTRODUCTION In elementary calculus you learned that differentiation and integration are *transforms*; this means, roughly speaking, that these operations transform a function into another function. For example, the function $f(x) = x^2$ is transformed, in turn, into a linear function and a family of cubic polynomial functions by the operations of differentiation and integration:

$$\frac{d}{dx}x^2 = 2x \qquad \text{and} \qquad \int x^2\,dx = \frac{1}{3}x^3 + c.$$

Moreover, these two transforms possess the **linearity property** that the transform of a linear combination of functions is a linear combination of the transforms. For α and β constants

$$\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x)$$

and

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

provided that each derivative and integral exists. In this section we will examine a special type of integral transform called the **Laplace transform.** In addition to possessing the linearity property the Laplace transform has many other interesting properties that make it very useful in solving linear initial-value problems.

INTEGRAL TRANSFORM If $f(x, y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable. For example, by holding y constant, we see that $\int_1^2 2xy^2\,dx = 3y^2$. Similarly, a definite integral such as $\int_a^b K(s, t)\,f(t)\,dt$ transforms a function f of the variable t into a function F of the variable s. We are particularly interested in an **integral transform,** where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_0^\infty K(s, t)\,f(t)\,dt$ is defined as a limit:

$$\int_0^\infty K(s, t)\,f(t)\,dt = \lim_{b\to\infty} \int_0^b K(s, t)\,f(t)\,dt. \tag{1}$$

If the limit in (1) exists, then we say that the integral exists or is **convergent;** if the limit does not exist, the integral does not exist and is **divergent.** The limit in (1) will, in general, exist for only certain values of the variable s.

A DEFINITION The function $K(s, t)$ in (1) is called the **kernel** of the transform. The choice $K(s, t) = e^{-st}$ as the kernel gives us an especially important integral transform.

DEFINITION 7.1.1 Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \tag{2}$$

is said to be the **Laplace transform** of f, provided that the integral converges.

When the defining integral (2) converges, the result is a function of s. In general discussion we shall use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace transform—for example,

$$\mathscr{L}\{f(t)\} = F(s), \qquad \mathscr{L}\{g(t)\} = G(s), \qquad \mathscr{L}\{y(t)\} = Y(s).$$

EXAMPLE 1 Applying Definition 7.1.1

Evaluate $\mathscr{L}\{1\}$.

SOLUTION From (2),

$$\begin{aligned}\mathscr{L}\{1\} &= \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt\\ &= \lim_{b\to\infty}\frac{-e^{-st}}{s}\Big|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}\end{aligned}$$

provided that $s > 0$. In other words, when $s > 0$, the exponent $-sb$ is negative, and $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$. ■

The use of the limit sign becomes somewhat tedious, so we shall adopt the notation $\big|_0^\infty$ as a shorthand for writing $\lim_{b\to\infty}(\ )\big|_0^b$. For example,

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \frac{-e^{-st}}{s}\Big|_0^\infty = \frac{1}{s}, \qquad s > 0.$$

At the upper limit, it is understood that we mean $e^{-st} \to 0$ as $t \to \infty$ for $s > 0$.

EXAMPLE 2 Applying Definition 7.1.1

Evaluate $\mathscr{L}\{t\}$.

SOLUTION From Definition 7.1.1 we have $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\,t\,dt$. Integrating by parts and using $\lim\limits_{t\to\infty} te^{-st} = 0$, $s > 0$, along with the result from Example 1, we obtain

$$\mathscr{L}\{t\} = \frac{-te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}.$$ ■

EXAMPLE 3 Applying Definition 7.1.1

Evaluate $\mathscr{L}\{e^{-3t}\}$.

SOLUTION From Definition 7.1.1 we have

$$\begin{aligned}\mathscr{L}\{e^{-3t}\} = \int_0^\infty e^{-st}e^{-3t}\,dt &= \int_0^\infty e^{-(s+3)t}\,dt\\ &= \frac{-e^{-(s+3)t}}{s+3}\Big|_0^\infty\\ &= \frac{1}{s+3}, \qquad s > -3.\end{aligned}$$

The result follows from the fact that $\lim_{t\to\infty} e^{-(s+3)t} = 0$ for $s + 3 > 0$ or $s > -3$. ■

EXAMPLE 4 Applying Definition 7.1.1

Evaluate $\mathscr{L}\{\sin 2t\}$.

SOLUTION From Definition 7.1.1 and integration by parts we have

$$\mathscr{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t\,dt = \frac{-e^{-st}\sin 2t}{s}\bigg|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt$$

$$= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt, \qquad s > 0$$

$$= \frac{2}{s}\Bigg[\underbrace{\frac{-e^{-st}\cos 2t}{s}\bigg|_0^\infty}_{\lim_{t\to\infty} e^{-st}\cos 2t = 0,\ s>0} - \frac{2}{s}\underbrace{\int_0^\infty e^{-st}\sin 2t\,dt}_{\text{Laplace transform of } \sin 2t}\Bigg]$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\mathscr{L}\{\sin 2t\}.$$

At this point we have an equation with $\mathscr{L}\{\sin 2t\}$ on both sides of the equality. Solving for that quantity yields the result

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2 + 4}, \qquad s > 0.$$ ■

$\mathscr{L}$ IS A LINEAR TRANSFORM For a linear combination of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\,dt = \alpha\int_0^\infty e^{-st}f(t)\,dt + \beta\int_0^\infty e^{-st}g(t)\,dt$$

whenever both integrals converge for $s > c$. Hence it follows that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s). \tag{3}$$

Because of the property given in (3), $\mathscr{L}$ is said to be a **linear transform.** For example, from Examples 1 and 2

$$\mathscr{L}\{1 + 5t\} = \mathscr{L}\{1\} + 5\mathscr{L}\{t\} = \frac{1}{s} + \frac{5}{s^2},$$

and from Examples 3 and 4

$$\mathscr{L}\{4e^{-3t} - 10\sin 2t\} = 4\mathscr{L}\{e^{-3t}\} - 10\mathscr{L}\{\sin 2t\} = \frac{4}{s+3} - \frac{20}{s^2+4}.$$

We state the generalization of some of the preceding examples by means of the next theorem. From this point on we shall also refrain from stating any restrictions on s; it is understood that s is sufficiently restricted to guarantee the convergence of the appropriate Laplace transform.

THEOREM 7.1.1 Transforms of Some Basic Functions

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$ **(c)** $\mathscr{L}\{e^{at}\} = \dfrac{1}{s - a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$ **(e)** $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$ **(g)** $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

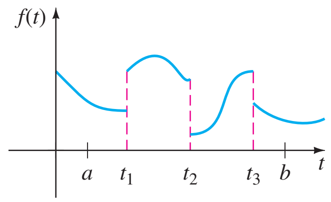


FIGURE 7.1.1 Piecewise continuous function

SUFFICIENT CONDITIONS FOR EXISTENCE OF $\mathscr{L}\{f(t)\}$ The integral that defines the Laplace transform does not have to converge. For example, neither $\mathscr{L}\{1/t\}$ nor $\mathscr{L}\{e^{t^2}\}$ exists. Sufficient conditions guaranteeing the existence of $\mathscr{L}\{f(t)\}$ are that f be piecewise continuous on $[0, \infty)$ and that f be of exponential order for $t > T$. Recall that a function f is **piecewise continuous** on $[0, \infty)$ if, in any interval $0 \le a \le t \le b$, there are at most a finite number of points t_k, $k = 1, 2, \ldots, n$ $(t_{k-1} < t_k)$ at which f has finite discontinuities and is continuous on each open interval (t_{k-1}, t_k). See Figure 7.1.1. The concept of **exponential order** is defined in the following manner.

DEFINITION 7.1.2 Exponential Order

A function f is said to be of **exponential order c** if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$.

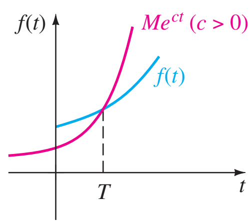


FIGURE 7.1.2 f is of exponential order c.

If f is an *increasing* function, then the condition $|f(t)| \le Me^{ct}$, $t > T$, simply states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}, where c is a positive constant. See Figure 7.1.2. The functions $f(t) = t$, $f(t) = e^{-t}$, and $f(t) = 2\cos t$ are all of exponential order $c = 1$ for $t > 0$, since we have, respectively,

$$|t| \le e^t, \quad |e^{-t}| \le e^t, \quad \text{and} \quad |2\cos t| \le 2e^t.$$

A comparison of the graphs on the interval $(0, \infty)$ is given in Figure 7.1.3.

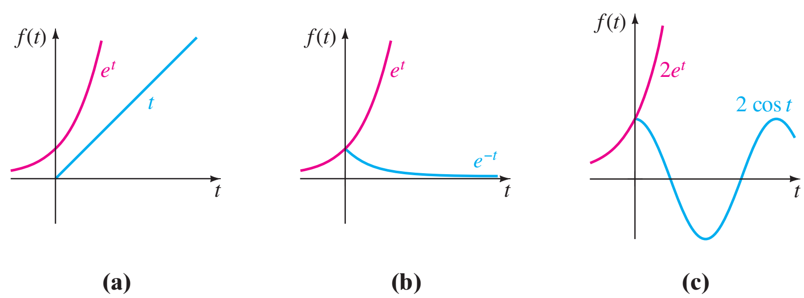


FIGURE 7.1.3 Three functions of exponential order $c = 1$

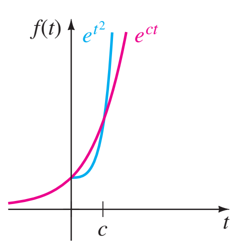


FIGURE 7.1.4 e^{t^2} is not of exponential order

A function such as $f(t) = e^{t^2}$ is not of exponential order, since, as shown in Figure 7.1.4, its graph grows faster than any positive linear power of e for $t > c > 0$.

A positive integral power of t is always of exponential order, since, for $c > 0$,

$$|t^n| \le Me^{ct} \quad \text{or} \quad \left|\frac{t^n}{e^{ct}}\right| \le M \quad \text{for } t > T$$

is equivalent to showing that $\lim_{t\to\infty} t^n/e^{ct}$ is finite for $n = 1, 2, 3, \ldots$. The result follows by n applications of L'Hôpital's Rule.

THEOREM 7.1.2 Sufficient Conditions for Existence

If f is piecewise continuous on $[0, \infty)$ and of exponential order c, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

PROOF By the additive interval property of definite integrals we can write

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt = I_1 + I_2.$$

The integral I_1 exists because it can be written as a sum of integrals over intervals on which $e^{-st}f(t)$ is continuous. Now since f is of exponential order, there exist constants c, $M > 0$, $T > 0$ so that $|f(t)| \le Me^{ct}$ for $t > T$. We can then write

$$|I_2| \le \int_T^\infty |e^{-st} f(t)|\,dt \le M \int_T^\infty e^{-st} e^{ct}\,dt = M \int_T^\infty e^{-(s-c)t}\,dt = M\frac{e^{-(s-c)T}}{s-c}$$

for $s > c$. Since $\int_T^\infty Me^{-(s-c)t}\,dt$ converges, the integral $\int_T^\infty |e^{-st}f(t)|\,dt$ converges by the comparison test for improper integrals. This, in turn, implies that I_2 exists for $s > c$. The existence of I_1 and I_2 implies that $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ exists for $s > c$. ■

EXAMPLE 5 Transform of a Piecewise Continuous Function

Evaluate $\mathscr{L}\{f(t)\}$ where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3. \end{cases}$

SOLUTION The function f, shown in Figure 7.1.5, is piecewise continuous and of exponential order for $t > 0$. Since f is defined in two pieces, $\mathscr{L}\{f(t)\}$ is expressed as the sum of two integrals:

$$\begin{aligned}\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt &= \int_0^3 e^{-st}(0)\,dt + \int_3^\infty e^{-st}(2)\,dt \\ &= 0 + \frac{2e^{-st}}{-s}\bigg|_3^\infty \\ &= \frac{2e^{-3s}}{s}, \qquad s > 0.\end{aligned}$$ ■

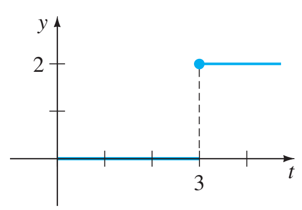


FIGURE 7.1.5 Piecewise continuous function

We conclude this section with an additional bit of theory related to the types of functions of s that we will, generally, be working with. The next theorem indicates that not every arbitrary function of s is a Laplace transform of a piecewise continuous function of exponential order.

THEOREM 7.1.3 Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $(0, \infty)$ and of exponential order and $F(s) = \mathscr{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

PROOF Since f is of exponential order, there exist constants γ, $M_1 > 0$, and $T > 0$ so that $|f(t)| \le M_1 e^{\gamma t}$ for $t > T$. Also, since f is piecewise continuous for $0 \le t \le T$, it is necessarily bounded on the interval; that is, $|f(t)| \le M_2 = M_2 e^{0t}$. If M denotes the maximum of the set $\{M_1, M_2\}$ and c denotes the maximum of $\{0, \gamma\}$, then

$$|F(s)| \le \int_0^\infty e^{-st}|f(t)|\,dt \le M\int_0^\infty e^{-st}e^{ct}\,dt = M\int_0^\infty e^{-(s-c)t}\,dt = \frac{M}{s-c}$$

for $s > c$. As $s \to \infty$, we have $|F(s)| \to 0$, and so $F(s) = \mathscr{L}\{f(t)\} \to 0$. ■

REMARKS

(*i*) Throughout this chapter we shall be concerned primarily with functions that are both piecewise continuous and of exponential order. We note, however, that these two conditions are sufficient but not necessary for the existence of a Laplace transform. The function $f(t) = t^{-1/2}$ is not piecewise continuous on the interval $[0, \infty)$, but its Laplace transform exists. See Problem 42 in Exercises 7.1.

(*ii*) As a consequence of Theorem 7.1.3 we can say that functions of s such as $F_1(s) = 1$ and $F_2(s) = s/(s + 1)$ are not the Laplace transforms of piecewise continuous functions of exponential order, since $F_1(s) \not\to 0$ and $F_2(s) \not\to 0$ as $s \to \infty$. But you should not conclude from this that $F_1(s)$ and $F_2(s)$ are *not* Laplace transforms. There are other kinds of functions.

EXERCISES 7.1

Answers to selected odd-numbered problems begin on page ANS-10.

In Problems 1–18 use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

1. $f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

2. $f(t) = \begin{cases} 4, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

3. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

4. $f(t) = \begin{cases} 2t + 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

5. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

6. $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \cos t, & t \ge \pi/2 \end{cases}$

7.

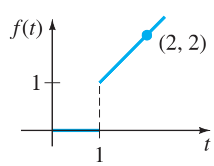


FIGURE 7.1.6 Graph for Problem 7

8.

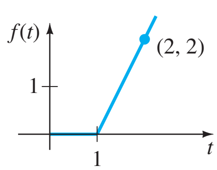


FIGURE 7.1.7 Graph for Problem 8

9.

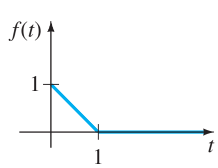


FIGURE 7.1.8 Graph for Problem 9

10.

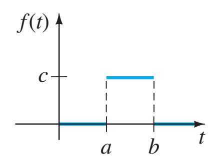


FIGURE 7.1.9 Graph for Problem 10

11. $f(t) = e^{t+7}$

12. $f(t) = e^{-2t-5}$

13. $f(t) = te^{4t}$

14. $f(t) = t^2e^{-2t}$

15. $f(t) = e^{-t}\sin t$

16. $f(t) = e^{t}\cos t$

17. $f(t) = t\cos t$

18. $f(t) = t\sin t$

In Problems 19–36 use Theorem 7.1.1 to find $\mathscr{L}\{f(t)\}$.

19. $f(t) = 2t^4$

20. $f(t) = t^5$

21. $f(t) = 4t - 10$

22. $f(t) = 7t + 3$

23. $f(t) = t^2 + 6t - 3$

24. $f(t) = -4t^2 + 16t + 9$

25. $f(t) = (t + 1)^3$

26. $f(t) = (2t - 1)^3$

27. $f(t) = 1 + e^{4t}$

28. $f(t) = t^2 - e^{-9t} + 5$

29. $f(t) = (1 + e^{2t})^2$

30. $f(t) = (e^{t} - e^{-t})^2$

31. $f(t) = 4t^2 - 5\sin 3t$

32. $f(t) = \cos 5t + \sin 2t$

33. $f(t) = \sinh kt$

34. $f(t) = \cosh kt$

35. $f(t) = e^{t}\sinh t$

36. $f(t) = e^{-t}\cosh t$

In Problems 37–40 find $\mathscr{L}\{f(t)\}$ by first using a trigonometric identity.

37. $f(t) = \sin 2t\cos 2t$

38. $f(t) = \cos^2 t$

39. $f(t) = \sin(4t + 5)$

40. $f(t) = 10\cos\left(t - \dfrac{\pi}{6}\right)$

41. One definition of the **gamma function** is given by the improper integral $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$, $\alpha > 0$.

(a) Show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

(b) Show that $\mathscr{L}\{t^\alpha\} = \dfrac{\Gamma(\alpha + 1)}{s^{\alpha+1}}, \quad \alpha > -1.$

42. Use the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and Problem 41 to find the Laplace transform of

(a) $f(t) = t^{-1/2}$ (b) $f(t) = t^{1/2}$ (c) $f(t) = t^{3/2}$.

Discussion Problems

43. Make up a function $F(t)$ that is of exponential order but where $f(t) = F'(t)$ is not of exponential order. Make up a function f that is not of exponential order but whose Laplace transform exists.

44. Suppose that $\mathscr{L}\{f_1(t)\} = F_1(s)$ for $s > c_1$ and that $\mathscr{L}\{f_2(t)\} = F_2(s)$ for $s > c_2$. When does

$$\mathscr{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)?$$

45. Figure 7.1.4 suggests, but does not prove, that the function $f(t) = e^{t^2}$ is not of exponential order. How does the observation that $t^2 > \ln M + ct$, for $M > 0$ and t sufficiently large, show that $e^{t^2} > Me^{ct}$ for any c?

46. Use part (c) of Theorem 7.1.1 to show that $\mathscr{L}\{e^{(a+ib)t}\} = \dfrac{s - a + ib}{(s - a)^2 + b^2}$, where a and b are real and $i^2 = -1$. Show how Euler's formula (page 134) can then be used to deduce the results

$$\mathscr{L}\{e^{at}\cos bt\} = \frac{s - a}{(s - a)^2 + b^2}$$

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s - a)^2 + b^2}.$$

47. Under what conditions is a linear function $f(x) = mx + b$, $m \neq 0$, a linear transform?

48. The proof of part (b) of Theorem 7.1.1 requires the use of mathematical induction. Show that if $\mathscr{L}\{t^{n-1}\} = (n - 1)!/s^n$ is *assumed* to be true, then $\mathscr{L}\{t^n\} = n!/s^{n+1}$ follows.

7.2 INVERSE TRANSFORMS AND TRANSFORMS OF DERIVATIVES

REVIEW MATERIAL

- Partial fraction decomposition
- See the *Student Resource and Solutions Manual*

INTRODUCTION In this section we take a few small steps into an investigation of how the Laplace transform can be used to solve certain types of equations for an unknown function. We begin the discussion with the concept of the inverse Laplace transform or, more precisely, the inverse of a Laplace transform $F(s)$. After some important preliminary background material on the Laplace transform of derivatives $f'(t), f''(t), \ldots$, we then illustrate how both the Laplace transform and the inverse Laplace transform come into play in solving some simple ordinary differential equations.

7.2.1 INVERSE TRANSFORMS

THE INVERSE PROBLEM If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathscr{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $f(t) = \mathscr{L}^{-1}\{F(s)\}$. For example, from Examples 1, 2, and 3 of Section 7.1 we have, respectively,

Transform	Inverse Transform
$\mathscr{L}\{1\} = \dfrac{1}{s}$	$1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$
$\mathscr{L}\{t\} = \dfrac{1}{s^2}$	$t = \mathscr{L}^{-1}\left\{\dfrac{1}{s^2}\right\}$
$\mathscr{L}\{e^{-3t}\} = \dfrac{1}{s + 3}$	$e^{-3t} = \mathscr{L}^{-1}\left\{\dfrac{1}{s + 3}\right\}$

We shall see shortly that in the application of the Laplace transform to equations we are not able to determine an unknown function $f(t)$ directly; rather, we are able to solve for the Laplace transform $F(s)$ of $f(t)$; but from that knowledge we ascertain f by computing $f(t) = \mathscr{L}^{-1}\{F(s)\}$. The idea is simply this: Suppose $F(s) = \dfrac{-2s + 6}{s^2 + 4}$ is a Laplace transform; find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$. We shall show how to solve this last problem in Example 2.

For future reference the analogue of Theorem 7.1.1 for the inverse transform is presented as our next theorem.

THEOREM 7.2.1 Some Inverse Transforms

$$\textbf{(a)}\ \ 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\textbf{(b)}\ \ t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\},\quad n = 1, 2, 3, \ldots \qquad \textbf{(c)}\ \ e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s - a}\right\}$$

$$\textbf{(d)}\ \ \sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2 + k^2}\right\} \qquad \textbf{(e)}\ \ \cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2 + k^2}\right\}$$

$$\textbf{(f)}\ \ \sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2 - k^2}\right\} \qquad \textbf{(g)}\ \ \cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2 - k^2}\right\}$$

In evaluating inverse transforms, it often happens that a function of s under consideration does not match *exactly* the form of a Laplace transform $F(s)$ given in a table. It may be necessary to "fix up" the function of s by multiplying and dividing by an appropriate constant.

EXAMPLE 1 Applying Theorem 7.2.1

Evaluate $\quad$ **(a)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ $\qquad$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + 7}\right\}$.

SOLUTION **(a)** To match the form given in part (b) of Theorem 7.2.1, we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) To match the form given in part (d) of Theorem 7.2.1, we identify $k^2 = 7$, so $k = \sqrt{7}$. We fix up the expression by multiplying and dividing by $\sqrt{7}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2 + 7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2 + 7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$ ■

$\mathscr{L}^{-1}$ IS A LINEAR TRANSFORM The inverse Laplace transform is also a linear transform; that is, for constants α and β

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\}, \tag{1}$$

where F and G are the transforms of some functions f and g. Like (2) of Section 7.1, (1) extends to any finite linear combination of Laplace transforms.

EXAMPLE 2 Termwise Division and Linearity

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

SOLUTION We first rewrite the given function of s as two expressions by means of termwise division and then use (1):

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\} \overset{\text{linearity and fixing up constants}}{=} -2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \qquad (2)$$

$$= -2\cos 2t + 3\sin 2t. \quad \leftarrow \text{parts (e) and (d) of Theorem 7.2.1 with } k = 2$$

PARTIAL FRACTIONS Partial fractions play an important role in finding inverse Laplace transforms. The decomposition of a rational expression into component fractions can be done quickly by means of a single command on most computer algebra systems. Indeed, some CASs have packages that implement Laplace transform and inverse Laplace transform commands. But for those of you without access to such software, we will review in this and subsequent sections some of the basic algebra in the important cases in which the denominator of a Laplace transform $F(s)$ contains distinct linear factors, repeated linear factors, and quadratic polynomials with no real factors. Although we shall examine each of these cases as this chapter develops, it still might be a good idea for you to consult either a calculus text or a current precalculus text for a more comprehensive review of this theory.

The following example illustrates partial fraction decomposition in the case when the denominator of $F(s)$ is factorable into *distinct linear factors.*

EXAMPLE 3 Partial Fractions: Distinct Linear Factors

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$.

SOLUTION There exist unique real constants A, B, and C so that

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$
$$= \frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}.$$

Since the denominators are identical, the numerators are identical:

$$s^2+6s+9 = A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2). \qquad (3)$$

By comparing coefficients of powers of s on both sides of the equality, we know that (3) is equivalent to a system of three equations in the three unknowns A, B, and C. However, there is a shortcut for determining these unknowns. If we set $s = 1$, $s = 2$, and $s = -4$ in (3), we obtain, respectively,

$$16 = A(-1)(5), \qquad 25 = B(1)(6), \qquad \text{and} \qquad 1 = C(-5)(-6),$$

and so $A = -\frac{16}{5}$, $B = \frac{25}{6}$, and $C = \frac{1}{30}$. Hence the partial fraction decomposition is

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1}+\frac{25/6}{s-2}+\frac{1/30}{s+4}, \qquad (4)$$

and thus, from the linearity of $\mathscr{L}^{-1}$ and part (c) of Theorem 7.2.1,

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}. \tag{5}$$

7.2.2 TRANSFORMS OF DERIVATIVES

TRANSFORM A DERIVATIVE As was pointed out in the introduction to this chapter, our immediate goal is to use the Laplace transform to solve differential equations. To that end we need to evaluate quantities such as $\mathscr{L}\{dy/dt\}$ and $\mathscr{L}\{d^2y/dt^2\}$. For example, if f' is continuous for $t \geq 0$, then integration by parts gives

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= -f(0) + s\mathscr{L}\{f(t)\}$$

or $$\mathscr{L}\{f'(t)\} = sF(s) - f(0). \tag{6}$$

Here we have assumed that $e^{-st}f(t) \to 0$ as $t \to \infty$. Similarly, with the aid of (6),

$$\mathscr{L}\{f''(t)\} = \int_0^\infty e^{-st}f''(t)\,dt = e^{-st}f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f'(t)\,dt$$

$$= -f'(0) + s\mathscr{L}\{f'(t)\}$$

$$= s[sF(s) - f(0)] - f'(0) \qquad \leftarrow \text{from (6)}$$

or $$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0). \tag{7}$$

In like manner it can be shown that

$$\mathscr{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0). \tag{8}$$

The recursive nature of the Laplace transform of the derivatives of a function f should be apparent from the results in (6), (7), and (8). The next theorem gives the Laplace transform of the nth derivative of f. The proof is omitted.

THEOREM 7.2.2 Transform of a Derivative

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

SOLVING LINEAR ODEs It is apparent from the general result given in Theorem 7.2.2 that $\mathscr{L}\{d^ny/dt^n\}$ depends on $Y(s) = \mathscr{L}\{y(t)\}$ and the $n - 1$ derivatives of $y(t)$ evaluated at $t = 0$. This property makes the Laplace transform ideally suited for solving linear initial-value problems in which the differential equation has *constant coefficients*. Such a differential equation is simply a linear combination of terms $y, y', y'', \ldots, y^{(n)}$:

$$a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0y = g(t),$$

$$y(0) = y_0,\ y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1},$$

where the a_i, $i = 0, 1, \ldots, n$ and $y_0, y_1, \ldots, y_{n-1}$ are constants. By the linearity property the Laplace transform of this linear combination is a linear combination of Laplace transforms:

$$a_n \mathscr{L}\left\{\frac{d^n y}{dt^n}\right\} + a_{n-1}\mathscr{L}\left\{\frac{d^{n-1} y}{dt^{n-1}}\right\} + \cdots + a_0\,\mathscr{L}\{y\} = \mathscr{L}\{g(t)\}. \tag{9}$$

From Theorem 7.2.2, (9) becomes

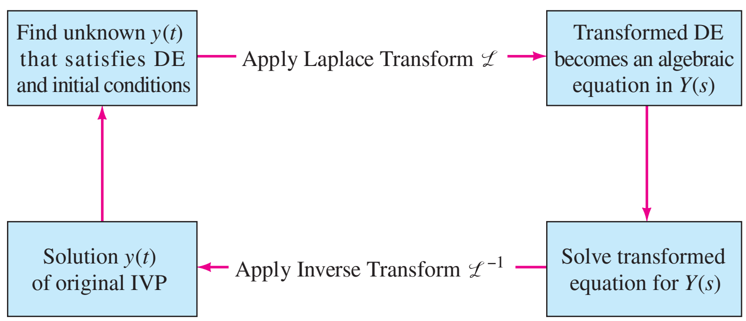
$$\begin{aligned} &a_n[s^n Y(s) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)] \\ &\quad + a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0) - \cdots - y^{(n-2)}(0)] + \cdots + a_0 Y(s) = G(s), \end{aligned} \tag{10}$$

where $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. In other words, *the Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in* $Y(s)$. If we solve the general transformed equation (10) for the symbol $Y(s)$, we first obtain $P(s)Y(s) = Q(s) + G(s)$ and then write

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}, \tag{11}$$

where $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$, $Q(s)$ is a polynomial in s of degree less than or equal to $n - 1$ consisting of the various products of the coefficients a_i, $i = 1, \ldots, n$ and the prescribed initial conditions $y_0, y_1, \ldots, y_{n-1}$, and $G(s)$ is the Laplace transform of $g(t)$.* Typically, we put the two terms in (11) over the least common denominator and then decompose the expression into two or more partial fractions. Finally, the solution $y(t)$ of the original initial-value problem is $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, where the inverse transform is done term by term.

The procedure is summarized in the following diagram.



The next example illustrates the foregoing method of solving DEs, as well as partial fraction decomposition in the case when the denominator of $Y(s)$ contains a *quadratic polynomial with no real factors.*

EXAMPLE 4 Solving a First-Order IVP

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13 \sin 2t, \quad y(0) = 6.$$

SOLUTION We first take the transform of each member of the differential equation:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 3\mathscr{L}\{y\} = 13\mathscr{L}\{\sin 2t\}. \tag{12}$$

*The polynomial $P(s)$ is the same as the nth-degree auxiliary polynomial in (12) in Section 4.3 with the usual symbol m replaced by s.

From (6), $\mathscr{L}\{dy/dt\} = sY(s) - y(0) = sY(s) - 6$, and from part (d) of Theorem 7.1.1, $\mathscr{L}\{\sin 2t\} = 2/(s^2 + 4)$, so (12) is the same as

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4} \qquad \text{or} \qquad (s+3)Y(s) = 6 + \frac{26}{s^2+4}.$$

Solving the last equation for $Y(s)$, we get

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} = \frac{6s^2+50}{(s+3)(s^2+4)}. \tag{13}$$

Since the quadratic polynomial $s^2 + 4$ does not factor using real numbers, its assumed numerator in the partial fraction decomposition is a linear polynomial in s:

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}.$$

Putting the right-hand side of the equality over a common denominator and equating numerators gives $6s^2 + 50 = A(s^2 + 4) + (Bs + C)(s + 3)$. Setting $s = -3$ then immediately yields $A = 8$. Since the denominator has no more real zeros, we equate the coefficients of s^2 and s: $6 = A + B$ and $0 = 3B + C$. Using the value of A in the first equation gives $B = -2$, and then using this last value in the second equation gives $C = 6$. Thus

$$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}.$$

We are not quite finished because the last rational expression still has to be written as two fractions. This was done by termwise division in Example 2. From (2) of that example,

$$y(t) = 8\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}.$$

It follows from parts (c), (d), and (e) of Theorem 7.2.1 that the solution of the initial-value problem is $y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$. ■

EXAMPLE 5 Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$.

SOLUTION Proceeding as in Example 4, we transform the DE. We take the sum of the transforms of each term, use (6) and (7), use the given initial conditions, use (c) of Theorem 7.2.1, and then solve for $Y(s)$:

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}. \tag{14}$$

The details of the partial fraction decomposition of $Y(s)$ have already been carried out in Example 3. In view of the results in (4) and (5) we have the solution of the initial-value problem

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$ ■

Examples 4 and 5 illustrate the basic procedure for using the Laplace transform to solve a linear initial-value problem, but these examples may appear to demonstrate a method that is not much better than the approach to such problems outlined in Sections 2.3 and 4.3–4.6. Don't draw any negative conclusions from only two examples. Yes, there is a lot of algebra inherent in the use of the Laplace transform, *but* observe that we do not have to use variation of parameters or worry about the cases and algebra in the method of undetermined coefficients. Moreover, since the method incorporates the prescribed initial conditions directly into the solution, there is no need for the separate operation of applying the initial conditions to the general solution $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$ of the DE to find specific constants in a particular solution of the IVP.

The Laplace transform has many operational properties. In the sections that follow we will examine some of these properties and see how they enable us to solve problems of greater complexity.

REMARKS

(*i*) The inverse Laplace transform of a function $F(s)$ may not be unique; in other words, it is possible that $\mathscr{L}\{f_1(t)\} = \mathscr{L}\{f_2(t)\}$ and yet $f_1 \neq f_2$. For our purposes this is not anything to be concerned about. If f_1 and f_2 are piecewise continuous on $[0, \infty)$ and of exponential order, then f_1 and f_2 are *essentially* the same. See Problem 44 in Exercises 7.2. However, if f_1 and f_2 are continuous on $[0, \infty)$ and $\mathscr{L}\{f_1(t)\} = \mathscr{L}\{f_2(t)\}$, then $f_1 = f_2$ on the interval.

(*ii*) This remark is for those of you who will be required to do partial fraction decompositions by hand. There is another way of determining the coefficients in a partial fraction decomposition in the special case when $\mathscr{L}\{f(t)\} = F(s)$ is a rational function of s and the denominator of F is a product of *distinct* linear factors. Let us illustrate by reexamining Example 3. Suppose we multiply both sides of the assumed decomposition

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4} \tag{15}$$

by, say, $s - 1$, simplify, and then set $s = 1$. Since the coefficients of B and C on the right-hand side of the equality are zero, we get

$$\left.\frac{s^2 + 6s + 9}{(s-2)(s+4)}\right|_{s=1} = A \qquad \text{or} \qquad A = -\frac{16}{5}.$$

Written another way,

$$\left.\frac{s^2 + 6s + 9}{\boxed{(s-1)}(s-2)(s+4)}\right|_{s=1} = -\frac{16}{5} = A,$$

where we have shaded, or *covered up*, the factor that canceled when the left-hand side was multiplied by $s - 1$. Now to obtain B and C, we simply evaluate the left-hand side of (15) while covering up, in turn, $s - 2$ and $s + 4$:

$$\left.\frac{s^2 + 6s + 9}{(s-1)\boxed{(s-2)}(s+4)}\right|_{s=2} = \frac{25}{6} = B$$

and

$$\left.\frac{s^2 + 6s + 9}{(s-1)(s-2)\boxed{(s+4)}}\right|_{s=-4} = \frac{1}{30} = C.$$

The desired decomposition (15) is given in (4). This special technique for determining coefficients is naturally known as the **cover-up method.**

(*iii*) In this remark we continue our introduction to the terminology of dynamical systems. Because of (9) and (10) the Laplace transform is well adapted to *linear* dynamical systems. The polynomial $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$ in (11) is the total coefficient of $Y(s)$ in (10) and is simply the left-hand side of the DE with the derivatives $d^k y/dt^k$ replaced by powers s^k, $k = 0, 1, \ldots, n$. It is usual practice to call the reciprocal of $P(s)$—namely, $W(s) = 1/P(s)$—the **transfer function** of the system and write (11) as

$$Y(s) = W(s)Q(s) + W(s)G(s). \tag{16}$$

In this manner we have separated, in an additive sense, the effects on the response that are due to the initial conditions (that is, $W(s)Q(s)$) from those due to the input function g (that is, $W(s)G(s)$). See (13) and (14). Hence the response $y(t)$ of the system is a superposition of two responses:

$$y(t) = \mathscr{L}^{-1}\{W(s)Q(s)\} + \mathscr{L}^{-1}\{W(s)G(s)\} = y_0(t) + y_1(t).$$

If the input is $g(t) = 0$, then the solution of the problem is $y_0(t) = \mathscr{L}^{-1}\{W(s)Q(s)\}$. This solution is called the **zero-input response** of the system. On the other hand, the function $y_1(t) = \mathscr{L}^{-1}\{W(s)G(s)\}$ is the output due to the input $g(t)$. Now if the initial state of the system is the zero state (all the initial conditions are zero), then $Q(s) = 0$, and so the only solution of the initial-value problem is $y_1(t)$. The latter solution is called the **zero-state response** of the system. Both $y_0(t)$ and $y_1(t)$ are particular solutions: $y_0(t)$ is a solution of the IVP consisting of the associated homogeneous equation with the given initial conditions, and $y_1(t)$ is a solution of the IVP consisting of the nonhomogeneous equation with zero initial conditions. In Example 5 we see from (14) that the transfer function is $W(s) = 1/(s^2 - 3s + 2)$, the zero-input response is

$$y_0(t) = \mathscr{L}^{-1}\left\{\frac{s+2}{(s-1)(s-2)}\right\} = -3e^t + 4e^{2t},$$

and the zero-state response is

$$y_1(t) = \mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s+4)}\right\} = -\frac{1}{5}e^t + \frac{1}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

Verify that the sum of $y_0(t)$ and $y_1(t)$ is the solution $y(t)$ in Example 5 and that $y_0(0) = 1$, $y_0'(0) = 5$, whereas $y_1(0) = 0$, $y_1'(0) = 0$.

EXERCISES 7.2

Answers to selected odd-numbered problems begin on page ANS-10.

7.2.1 INVERSE TRANSFORMS

In Problems 1–30 use appropriate algebra and Theorem 7.2.1 to find the given inverse Laplace transform.

1. $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$

2. $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$

3. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\}$

4. $\mathscr{L}^{-1}\left\{\left(\frac{2}{s} - \frac{1}{s^3}\right)^2\right\}$

5. $\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$

6. $\mathscr{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\}$

7. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2}\right\}$

8. $\mathscr{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^5} - \frac{1}{s+8}\right\}$

9. $\mathscr{L}^{-1}\left\{\frac{1}{4s+1}\right\}$

10. $\mathscr{L}^{-1}\left\{\frac{1}{5s-2}\right\}$

11. $\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\}$

12. $\mathscr{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$

13. $\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$

14. $\mathscr{L}^{-1}\left\{\frac{1}{4s^2+1}\right\}$

15. $\mathscr{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$

16. $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\}$

17. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + 3s}\right\}$

18. $\mathscr{L}^{-1}\left\{\dfrac{s + 1}{s^2 - 4s}\right\}$

19. $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 + 2s - 3}\right\}$

20. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + s - 20}\right\}$

21. $\mathscr{L}^{-1}\left\{\dfrac{0.9s}{(s - 0.1)(s + 0.2)}\right\}$

22. $\mathscr{L}^{-1}\left\{\dfrac{s - 3}{(s - \sqrt{3})(s + \sqrt{3})}\right\}$

23. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s - 2)(s - 3)(s - 6)}\right\}$

24. $\mathscr{L}^{-1}\left\{\dfrac{s^2 + 1}{s(s - 1)(s + 1)(s - 2)}\right\}$

25. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3 + 5s}\right\}$

26. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s + 2)(s^2 + 4)}\right\}$

27. $\mathscr{L}^{-1}\left\{\dfrac{2s - 4}{(s^2 + s)(s^2 + 1)}\right\}$

28. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4 - 9}\right\}$

29. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2 + 1)(s^2 + 4)}\right\}$

30. $\mathscr{L}^{-1}\left\{\dfrac{6s + 3}{s^4 + 5s^2 + 4}\right\}$

7.2.2 TRANSFORMS OF DERIVATIVES

In Problems 31–40 use the Laplace transform to solve the given initial-value problem.

31. $\dfrac{dy}{dt} - y = 1, \quad y(0) = 0$

32. $2\dfrac{dy}{dt} + y = 0, \quad y(0) = -3$

33. $y' + 6y = e^{4t}, \quad y(0) = 2$

34. $y' - y = 2\cos 5t, \quad y(0) = 0$

35. $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

36. $y'' - 4y' = 6e^{3t} - 3e^{-t}, \quad y(0) = 1, \quad y'(0) = -1$

37. $y'' + y = \sqrt{2}\sin\sqrt{2}t, \quad y(0) = 10, \quad y'(0) = 0$

38. $y'' + 9y = e^t, \quad y(0) = 0, \quad y'(0) = 0$

39. $2y''' + 3y'' - 3y' - 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

40. $y''' + 2y'' - y' - 2y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

The inverse forms of the results in Problem 46 in Exercises 7.1 are

$$\mathscr{L}^{-1}\left\{\frac{s - a}{(s - a)^2 + b^2}\right\} = e^{at}\cos bt$$

$$\mathscr{L}^{-1}\left\{\frac{b}{(s - a)^2 + b^2}\right\} = e^{at}\sin bt.$$

In Problems 41 and 42 use the Laplace transform and these inverses to solve the given initial-value problem.

41. $y' + y = e^{-3t}\cos 2t, \quad y(0) = 0$

42. $y'' - 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 3$

Discussion Problems

43. (a) With a slight change in notation the transform in (6) is the same as

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0).$$

With $f(t) = te^{at}$, discuss how this result in conjunction with (c) of Theorem 7.1.1 can be used to evaluate $\mathscr{L}\{te^{at}\}$.

(b) Proceed as in part (a), but this time discuss how to use (7) with $f(t) = t\sin kt$ in conjunction with (d) and (e) of Theorem 7.1.1 to evaluate $\mathscr{L}\{t\sin kt\}$.

44. Make up two functions f_1 and f_2 that have the same Laplace transform. Do not think profound thoughts.

45. Reread *Remark* (*iii*) on page 269. Find the zero-input and the zero-state response for the IVP in Problem 36.

46. Suppose $f(t)$ is a function for which $f'(t)$ is piecewise continuous and of exponential order c. Use results in this section and Section 7.1 to justify

$$f(0) = \lim_{s\to\infty} sF(s),$$

where $F(s) = \mathscr{L}\{f(t)\}$. Verify this result with $f(t) = \cos kt$.

7.3 OPERATIONAL PROPERTIES I

REVIEW MATERIAL

- Keep practicing partial fraction decomposition
- Completion of the square

INTRODUCTION It is not convenient to use Definition 7.1.1 each time we wish to find the Laplace transform of a function $f(t)$. For example, the integration by parts involved in evaluating, say, $\mathscr{L}\{e^t t^2 \sin 3t\}$ is formidable, to say the least. In this section and the next we present several labor-saving operational properties of the Laplace transform that enable us to build up a more extensive list of transforms (see the table in Appendix III) without having to resort to the basic definition and integration.